RANDOM CHORDS OF A SPHERE

BY

DAVID BERENGUT

TECHNICAL REPORT NO. 191
MARCH 30, 1972

PREPARED UNDER CONTRACT N00014-67-A-0112-0053
(NR-042-267)
OFFICE OF NAVAL RESEARCH
Herbert Solomon, Project Director

DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

UNCLASSIFIED

Security Classification

DOCUMENT CONTROL DATA - R&D

(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)

1. ORIGINATING ACTIVITY (Corporate author)

STANFORD UNIVERSITY
STATISTICS DEPARTMENT
STANFORD, CALIFORNIA

2a. REPORT SECURITY CLASSIFICATION

2b. GROUP

3. REPORT TITLE

RANDOM CHORDS OF A SPHERE

4. DESCRIPTIVE NOTES (Type of report and inclusive dates)

TECHNICAL REPORT

5. AUTHOR(S) (Last name, first name, initial)

BERENGUT, DAVID

6. REPORT DATE	7a. TOTAL NO. OF PAGES	7b. NO. OF REFS
March 30, 1972	20	8

8a. CONTRACT OR GRANT NO.

N00014-67-A-0112-0053

b. PROJECT NO.

NR-042-267

c.

d.

9a. ORIGINATOR'S REPORT NUMBER(S)

TECHNICAL REPORT NO. 191

9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report)

10. AVAILABILITY/LIMITATION NOTICES

DISTRIBUTION OF THIS DOCUMENT IS UNLIMITED

11. SUPPLEMENTARY NOTES

12. SPONSORING MILITARY ACTIVITY

Statistics & Probability Program
Office of Naval Research
Arlington, VA

13. ABSTRACT

Eight models of randomness for chords of a unit sphere are considered, and the distribution, mean, and variance of the chord length are obtained for each model.

DD FORM 1 JAN 64 1473

UNCLASSIFIED

Security Classification

UNCLASSIFIED

Security Classification

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Chords						
Sphere						

INSTRUCTIONS

1. ORIGINATING ACTIVITY: Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization (*corporate author*) issuing the report.

2a. REPORT SECURITY CLASSIFICATION: Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. GROUP: Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. REPORT TITLE: Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. DESCRIPTIVE NOTES: If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. AUTHOR(S): Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. REPORT DATE: Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. TOTAL NUMBER OF PAGES: The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. NUMBER OF REFERENCES: Enter the total number of references cited in the report.

8a. CONTRACT OR GRANT NUMBER: If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. PROJECT NUMBER: Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. ORIGINATOR'S REPORT NUMBER(S): Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. OTHER REPORT NUMBER(S): If the report has been assigned any other report numbers (*either by the originator or by the sponsor*), also enter this number(s).

10. AVAILABILITY/LIMITATION NOTICES: Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

(1) "Qualified requesters may obtain copies of this report from DDC."

(2) "Foreign announcement and dissemination of this report by DDC is not authorized."

(3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through ________________."

(4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through ________________."

(5) "All distribution of this report is controlled. Qualified DDC users shall request through ________________."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. SUPPLEMENTARY NOTES: Use for additional explanatory notes.

12. SPONSORING MILITARY ACTIVITY: Enter the name of the departmental project office or laboratory sponsoring (*paying for*) the research and development. Include address.

13. ABSTRACT: Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as *(TS)*, *(S)*, *(C)*, or *(U)*.

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. KEY WORDS: Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1 JAN 64 1473 (BACK)

Unclassified

Security Classification

RANDOM CHORDS OF A SPHERE

by

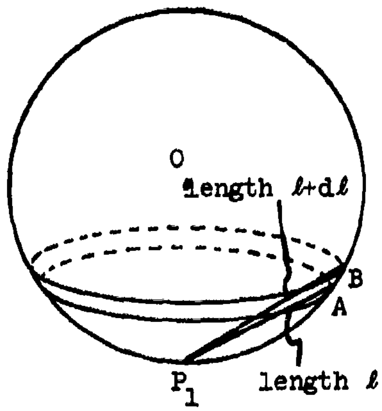
David Berengut

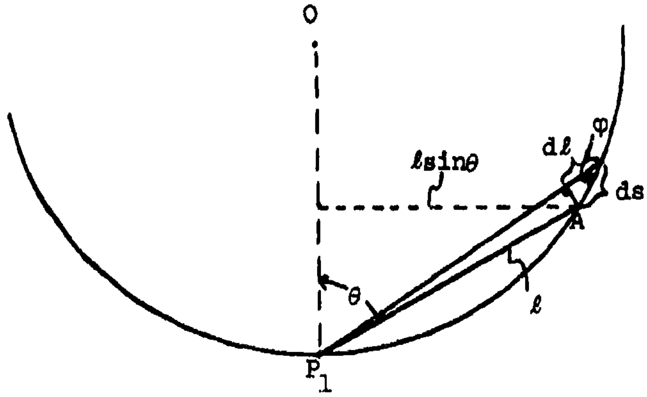
Random chords (also referred to as random secants) of convex regions have been examined by a number of investigators. In connection with acoustics, Jäger [5], Bate and Pillow [1], and Kingman [7] studied mean free paths in convex regions in E^3. Horowitz [4] obtained the distribution of the chord length for a rectangle in the case where the chord is radiated in a random direction from a point uniformly distributed on the boundary. Coleman [2] studied the distribution of random chords of a circle, rectangle, and cube under several randomness models. Kingman [8] considered random chords of a convex body in E^n, and obtained relations between three different measures. In connection with the question of whether chromosome pairs are randomly distributed with respect to each other during cell mitosis, David and Fix [3] examined the intersections of random chords of a circle under several randomness models.

In this paper, we will consider the distribution of the length of a random chord of a sphere, under eight randomness models, and obtain the mean and variance in each case. We assume throughout that the sphere is of unit radius, and we let L be the random variable denoting chord length, with density $f(\ell)$, $0 \leq \ell \leq 2$.

Model 1: Chord joining two points independently uniformly distributed on surface of sphere.

We can let the position P_1 of the first point be arbitrary. Then the $\Pr\{\ell < L < \ell + d\ell\}$ is $\frac{1}{4\pi}$ times the area of the circular band in the following figure:





Cross-section through P_1,A,B

Now the width of the band is $ds = \sec\varphi \, d\ell$. But φ is the angle between P_1B and the tangent to the circle at B. Hence by elementary geometry, φ is equal to the angle subtended on the upper part of the circle by the chord P_1B. But this angle is just $\frac{\pi}{2} - \theta$. Hence $\varphi = \frac{\pi}{2} - \theta$, and therefore $ds = \csc\theta d\ell$. Now the area of the band is ds times its circumference, which is $2\pi \cdot \ell\sin\theta$.

$\therefore$ Area of band $= \csc\theta\, d\ell \cdot 2\pi\, \ell \sin\theta = 2\pi\ell\, d\ell$

$\therefore \Pr\{\ell < L < \ell + d\ell\} = \frac{1}{4\pi} \cdot 2\pi\, \ell\, d\ell = \frac{\ell}{2}\, d\ell$

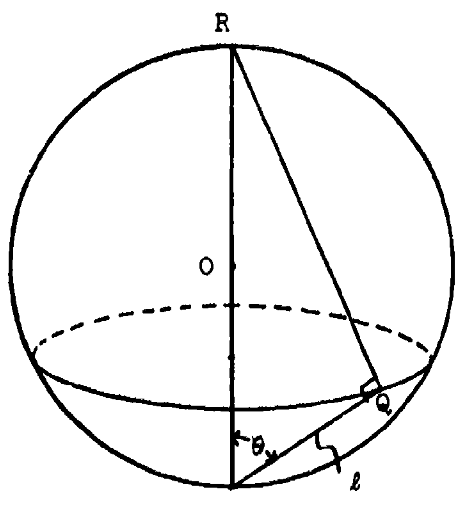
$\therefore f(\ell) = \ell/2\ , \quad 0 \le \ell \le 2$

$$E(\ell) = \int_0^2 \ell^2/2\ d\ell = [\frac{\ell^3}{6}]_0^2 = \frac{4}{3} \qquad (1)$$

$$\mathrm{Var}(\ell) = \int_0^2 \ell^3/2\ d\ell - \frac{16}{9} = [\frac{\ell^4}{8}]_0^2 - \frac{16}{9} = \frac{2}{9}\ .$$

<u>Model 2</u>: <u>Chord in random direction from point uniformly distributed on surface</u>.

Let the position of the point be P. Let θ be the angle between the random chord through P and the diameter through P.



Since the chord PQ is randomly directed, we know from a well-known result that the density of θ is proportional to $\sin\theta$, i.e., θ has density

$g(\theta) = \sin\theta,\ 0 \le \theta \le \frac{\pi}{2}$. Now triangle PQR is a right triangle with hypotenuse of length 2, hence $\ell = \cos\theta$. Thus $\left|\frac{d\ell}{d\theta}\right| = 2\sin\theta$. The density of ℓ is thus $f(\ell) = g(\theta)/\left|\frac{d\ell}{d\theta}\right| = \frac{\sin\theta}{2\sin\theta} = \frac{1}{2}$. i.e.,

$$f(\ell) = \frac{1}{2},\quad 0 \le \ell \le 2\ . \tag{2}$$

$$E(\ell) = 1,\quad \mathrm{Var}(\ell) = \frac{1}{3}\ .$$

<u>Model 3</u>: <u>Chord in random direction, distance from center uniformly distributed</u>.

Let r be the distance of the chord from the center of the circle. Then

$$\ell = 2\sqrt{1-r^2}\ ;\quad \left|\frac{d\ell}{dr}\right| = \left|\frac{2(-2r)}{2\sqrt{1-r^2}}\right| = \frac{2r}{\sqrt{1-r^2}}\ .$$

Since r is uniformly distributed on $(0,1)$,

$$f(\ell) = \frac{1}{\dfrac{2r}{\sqrt{1-r^2}}} = \frac{\sqrt{1-r^2}}{2r} = \frac{\ell/2}{2\sqrt{1-\ell^2/4}} = \frac{\ell}{2\sqrt{4-\ell^2}}\ ,\quad 0 \le \ell \le 2\ \ldots \tag{3}$$

$$E(\ell) = \frac{1}{2}\int_0^2 \frac{\ell^2}{\sqrt{4-\ell^2}}\,d\ell$$

Let $u = \frac{4-\ell^2}{4}$. Then $\ell = 2\sqrt{1-u}$, $du = -\frac{\ell}{2}\,d\ell$

$$\therefore\ E(\ell) = \frac{1}{2}\int_1^0 \frac{4(1-u)}{2\sqrt{u}}\ \frac{-2}{2\sqrt{1-u}}\,du$$

$$= \int_0^1 u^{-1/2}(1-u)^{1/2}du$$

$$= \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{\Gamma(2)} = \frac{1}{2}\sqrt{\pi}\sqrt{\pi} = \frac{\pi}{2}$$

$$E(\ell^2) = 2\int_0^1 u^{-1/2}(1-u)\,du = \frac{2\Gamma(\frac{1}{2})u(2)}{\Gamma(\frac{5}{2})} = \frac{2\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{8}{3}$$

$$\therefore \mathrm{Var}(\ell) = \frac{8}{3} - \frac{\pi^2}{4}\,.$$

Model 4: __**Center of chord uniformly distributed inside sphere, random direction.**__

Let r be the distance of the chord (i.e., the center of the chord) from the center of the sphere.

$$\Pr(r_0 \le r \le r_0+dr) = \Pr(r \le r_0+dr) - \Pr(r \le r_0)$$

$$= \frac{1}{\frac{4}{3}\pi}\left(\frac{4}{3}\pi(r_0+dr)^3\right) - \frac{1}{\frac{4}{3}\pi}\left(\frac{4}{3}\pi r_0^3\right)$$

$$= (r_0+dr)^3 - r_0^3$$

$$= 3r_0^2\,dr + o(dr)$$

Hence r has density $g(r) = 3r^2$, $0 \le r \le 1$. (4)

Now $\ell = 2\sqrt{1-r^2}$, $r = \sqrt{4-\ell^2/2}$, $\left|\frac{d\ell}{dr}\right| = \frac{2r}{\sqrt{1-r^2}}$. Hence the density of ℓ is given by

$$f(\ell) = 3r^2\frac{\sqrt{1-r^2}}{2r} = \frac{3}{2}r\sqrt{1-r^2} = \frac{3}{2}\frac{\sqrt{4-\ell^2}}{2}\cdot\frac{\ell}{2} = \frac{3}{8}\ell\sqrt{4-\ell^2}\,,$$

$$0 \le \ell \le 2\,. \qquad (5)$$

$$E(\ell) = \frac{3}{8}\int_0^2 \ell^2\sqrt{4-\ell^2}\,d\ell$$

As before, letting $u = \frac{4-\ell^2}{4}$, we get

$$E(\ell) = \frac{3}{8}\int_0^1 4(1-u) \cdot 2u^{1/2} \cdot \frac{2}{2(1-u)^{1/2}}\,du = 3\int_0^1 u^{1/2}(1-u)^{1/2}du$$

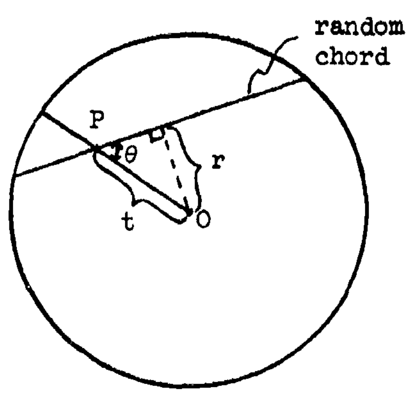
$$= \frac{3\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{\Gamma(3)} = \frac{3 \cdot \frac{\pi}{4}}{2} = \frac{3\pi}{8}.$$

Similarly, $E(\ell^2) = 6\int_0^1 u^{1/2}(1-u)du = \frac{6\Gamma(\frac{3}{2})\Gamma(2)}{\Gamma(\frac{7}{2})} = \frac{6 \cdot \frac{1}{2}\sqrt{\pi}}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}} = \frac{8}{5}$

$$\therefore \operatorname{Var}(\ell) = \frac{8}{5} - \frac{9\pi^2}{64}.$$

<u>Model 5: Chord in random direction through point uniformly distributed inside sphere.</u>

Let t be the distance of the random point P from the center of the sphere. Let θ be the acute angle between the chord and the radius through P. Then the distance from the center to the chord is $r = t \sin\theta$. The following figure, which gives the cross-section through the sphere containing the chord and the center of the sphere, illustrates this.



Now, since the chord through P has random direction, we know that θ has density $g(\theta) = \sin\theta,\ 0 \leq \theta \leq \frac{\pi}{2}$. Let t be given. Then $dr = t\cos\theta\, d\theta$. Thus the conditional density of r, given t is

$$h(r|t) = \sin\theta \frac{1}{t\cos\theta} = \frac{\tan\theta}{t} = \frac{r}{t\sqrt{t^2-r^2}},\ 0 \leq r \leq t \qquad (6)$$

Now since P is uniformly distributed inside the sphere, t has density $g(t) = 3t^2,\ 0 \leq t \leq 1$, as was demonstrated previously in Model 4. Hence the density of r is

$$\begin{aligned} h(r) &= \int_0^1 h(r|t)g(t)dt \\ &= \int_r^1 \frac{r}{t\sqrt{t^2-r^2}}\, 3t^2 dt \\ &= 3r\int_r^1 \frac{t}{\sqrt{t^2-r^2}}\, dt \\ &= \frac{3r}{2}\int_0^{1-r^2} \frac{du}{\sqrt{u}} = \frac{3r}{2}\left[2u^{1/2}\right]_0^{1-r^2} = 3r\sqrt{1-r^2} \end{aligned}$$

$$\text{i.e.,}\quad h(r) = 3r\sqrt{1-r^2},\ 0 \leq r \leq 1 \qquad (7)$$

Now

$$\ell = 2\sqrt{1-r^2}\ , \quad \left|\frac{d\ell}{dr}\right| = \frac{2r}{\sqrt{1-r^2}}\ .$$

$$\therefore\ f(\ell) = 3r\sqrt{1-r^2}\cdot\frac{\sqrt{1-r^2}}{2r} = \frac{3}{2}(1-r^2) = \frac{3}{8}\ell^2\ , \quad 0 \le \ell \le 2 \qquad (8)$$

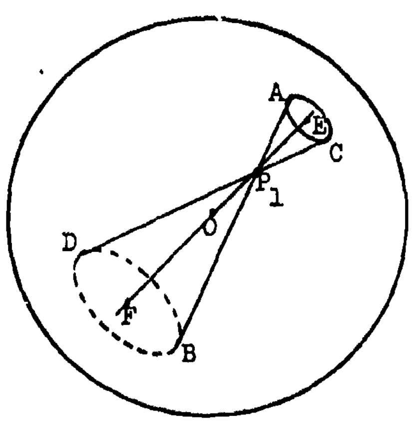
$$E(\ell) = \frac{3}{8}\int_0^2 \ell^3 d\ell = \frac{3}{32}\cdot 16 = \frac{3}{2}\ .$$

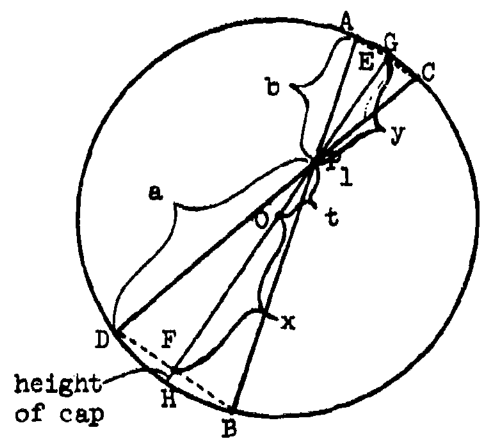
$$E(\ell^2) = \frac{3}{8}\int_0^2 \ell^4 d\ell = \frac{3}{40}\cdot 32 = \frac{12}{5}$$

$$\therefore\ \mathrm{Var}(\ell) = \frac{12}{5} - \frac{9}{4} = \frac{3}{20}\ .$$

<u>Model 6</u>: <u>Chord through two points independently uniformly distributed inside sphere</u>.

Let the position of one of the points P_1 be at a distance t from the center. We will first condition on t. Consider $P(L \ge \ell|t)$, for $2\sqrt{1-t^2} \le \ell \le 2$. This is just $\frac{1}{4\pi}$ times the volume of a pair of conical regions inside the sphere, with common axis the diameter through P_1. The following figure illustrates the region.

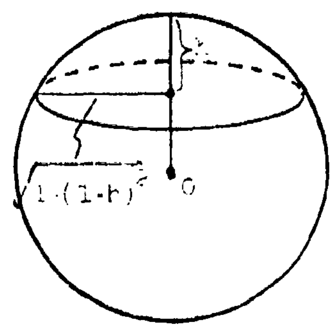




Cross-section Containing A, B, D, C

We have $|P_1C| = |P_1A| = b$, $|P_1B| = |P_1D| = a$, $a+b = \ell$. Also, $|P_1E| = y$, $|OF| = x$. (If F lies on the same side of O as P_1, then we take x to be negative). It is clear that $L \geq \ell$ if and only if the second point P_2 lies inside the regions bounded by the sides of the cones. Now the region above P_1 (having A and C as boundary points) consists of a cone with flat base of height y, plus a spherical cap of height $1-t-y$. Similarly, the region below P_1 (having D and B as boundary points) consists of a cone of height $x+t$ and a spherical cap of height $1-x$. We want to obtain the volume of these regions in terms of t and ℓ.

Consider a spherical cap of height h, $0 \leq h \leq 1$. Its base is a circle of radius $\sqrt{1-(1-h)^2}$.



Now, clearly, the volume of the cap is given by

$$V(h) = \int_0^{2\pi} \int_0^{\sqrt{1-(1-h)^2}} \left[\sqrt{1-r^2} - (1-h)\right] r \, dr \, d\theta .$$

(cont.)

$$= 2\pi \quad \frac{1}{2}\left[\frac{(1-r^2)^{3/2}}{3/2}\right]_{r=\sqrt{1-(1-h)^2}}^{r=0} - 2\pi(1-h)\,\frac{1-(1-h)^2}{2}$$

$$= \pi\left[\frac{2}{3}\,(1-(1-h)^3) - (1-h) + (1-h)^3\right]$$

$$= \pi\left[\frac{2}{3} - (1-h) + \frac{1}{3}\,(1-h)^3\right] \quad , \quad 0 \leq h \leq 1 \quad .$$

Now for $1 \leq h \leq 2$, $V(h) = \frac{4}{3}\pi - V(2-h) = \pi\left(\frac{2}{3} - (1-h) + \frac{1}{3}\,(1-h)^3\right)$

Thus

$$V(h) = \pi\left(\frac{2}{3} - (1-h) + \frac{1}{3}\,(1-h)^3\right), \quad 0 \leq h \leq 2 \tag{9}$$

Hence the volume of the upper cap is

$$V_1 = \pi\left(\frac{2}{3} - (y+t) + \frac{(y+t)^3}{3}\right) \tag{10}$$

Now, for the lower cap, since we take x to be negative for $h > 1$, we have $1-h=x$ in all cases. Hence the volume of the lower cap is

$$V_2 = \pi\left(\frac{2}{3} - x + \frac{x^3}{3}\right) \tag{11}$$

Now the upper cone has base radius $\sqrt{b^2-y^2}$ and height y, hence its volume is

$$V_3 = \frac{1}{3}\,\pi\,(b^2-y^2)y \tag{12}$$

Similarly, the lower cone has volume

$$V_4 = \frac{1}{3}\pi\,(1-x^2)(t+x) \tag{13}$$

Adding (10) - (13), we get

$$V=V_1+V_2+V_3+V_4=\pi\left[\frac{4}{3}+(\frac{b^2}{3}+t^2-1)y+ty^2-\frac{2}{3}x-\frac{t}{3}x^2-\frac{2t}{3}+\frac{t^3}{3}\right] \tag{14}$$

We want to express V in terms of t and ℓ. To do this, we have to find expressions for $a,b,x,$ and y in terms of t and ℓ.

Now, from elementary geometry, we know that for two intersecting chords of a circle, the product of the segments of one is equal to the product of the segments of the other. Applying this to chords AB and GH, we get $ab = (1-t)(1+t)$. Now using the relation $a + b = \ell$, we can solve a quadratic equation to obtain the relations.

$$a = \frac{\ell+\sqrt{\ell^2-4(1-t^2)}}{2} \tag{15}$$

$$b = \frac{\ell-\sqrt{\ell^2-4(1-t^2)}}{2} \tag{16}$$

We also have the relations $b^2-y^2 = 1-(y+t)^2$ and $1-x^2 = a^2-(t+x)^2$, from which we obtain the following expressions for x and y:

$$x = \frac{a^2-t^2-1}{2t} \tag{17}$$

$$y = \frac{1-t^2-b^2}{2t} \tag{18}$$

Using (15) and (16), we obtain finally

$$x = \frac{\ell^2-4+\ell\sqrt{\ell^2-4(1-t^2)}}{4t} \tag{19}$$

$$y = \frac{4-\ell^2+\ell\sqrt{\ell^2-4(1-t^2)}}{4t} - t \tag{20}$$

Substituting these expressions into (14), after considerable simplification we obtain

$$V = \pi\left(\frac{4}{3} - \frac{\ell^3\sqrt{\ell^2-4(1-t^2)}}{12t}\right), \quad 2\sqrt{1-t^2} \le \ell \le 2 \tag{21}$$

Now since $P(L > \ell|t) = \left(\frac{4\pi}{3}\right)^{-1} V$, for $2\sqrt{1-t^2} \le \ell \le 2$, we have:

$$P(L > \ell|t) = \begin{cases} 1 - \dfrac{\ell^3\sqrt{\ell^2-4(1-t^2)}}{16t}, & 2\sqrt{1-t^2} \le \ell \le 2 \\ 1, & 0 \le \ell \le 2\sqrt{1-t^2} \end{cases} \tag{22}$$

Now t, being the distance of P_1 from the center O, has density $f(t) = 3t^2$, as we saw before in (4). Hence

$$P(L > \ell) = \int_0^1 P(L > \ell | t) 3t^2 dt$$

$$= \int_0^1 3t^2 dt - \int_{\frac{\sqrt{4-\ell^2}}{2}}^1 \frac{\ell^3 \sqrt{\ell^2 - 4(1-t^2)}}{16t} \, 3t^2 dt$$

$$= 1 - \frac{3\ell^3}{16} \int_{\frac{\sqrt{4-\ell^2}}{2}}^1 t \sqrt{\ell^2 - 4(1-t^2)} \, dt$$

$$= 1 - \frac{3\ell^3}{16} \cdot \frac{1}{8} \cdot \frac{2}{3} \left[\left\{ \ell^2 - 4(1-t^2) \right\}^{3/2} \right]_{t = \sqrt{4-\ell^2}/2}^{t=1}$$

$$= 1 - \frac{\ell^6}{64} \tag{23}$$

Hence $P(L \leq \ell) = \frac{\ell^6}{64}$, $0 \leq \ell \leq 2$, and the density of L is

$$f(\ell) = \frac{3}{32} \ell^5, \quad 0 \leq \ell \leq 2 \tag{24}$$

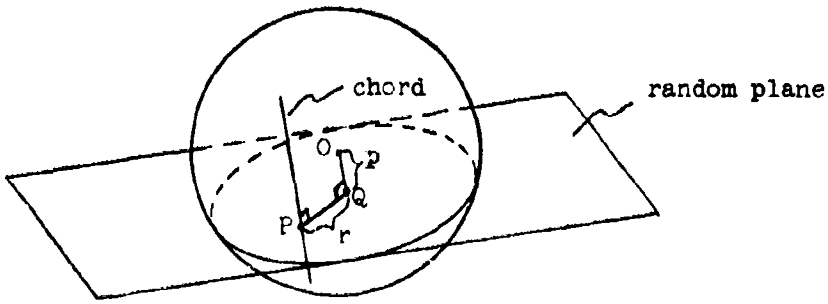
$$E(\ell) = \frac{3}{32} \int_0^2 \ell^6 d\ell = \frac{3}{32} \left(\frac{2^7}{2}\right) = \frac{12}{7}$$

$$E(\ell^2) = \frac{3}{32} \int_0^2 \ell^7 d\ell = \frac{3}{32} \left(\frac{2^8}{8}\right) = 3$$

$$\therefore \ \mathrm{Var}\,\ell = 3 - \frac{144}{49} = \frac{3}{49}$$

Model 7: Chord normal to random intersecting plane, through point uniformly distributed inside circle of intersection.

By random planes we mean planes with measure invariant under rotations, translations and reflections. We are interested in the conditional space of random planes which intersect the sphere. Let p be the distance of the random intersecting plane from the center of the sphere. The random chord is obtained by choosing a point P having uniform distribution in the circle of intersection of the plane with the sphere, and drawing the normal to the plane through that point. Let r be the distance of the point P from the center Q of the circle of intersection, $0 \le r \le \sqrt{1-p^2}$. (See figure.)



Let p be given. Then

$$P(r_0 \le r \le r_0+dr) = \frac{1}{\pi(1-p^2)} \left[\pi(r_0+dr)^2 - \pi r_0^2\right], \quad 0 \le r_0 \le \sqrt{1-p^2} .$$

$$= \frac{2r_0}{1-p^2}\, dr + o(dr)$$

Hence the conditional density of r given p is

$$g(r|p) = \frac{2r}{1-p^2}\,, \quad 0 \le r \le 1 - p^2 \tag{25}$$

Now since OQ and the chord are both normal to the same plane, the distance from the chord to the center O is r. Hence the length of the chord is $\ell = 2\sqrt{1-r^2}$, $\left|\frac{d\ell}{dr}\right| = \frac{2r}{\sqrt{1-r^2}}$. Making the transformation to ℓ, we obtain the conditional density of ℓ:

$$f(\ell|p) = \frac{2r}{1-p^2} \cdot \frac{\sqrt{1-r^2}}{2r} = \frac{\sqrt{1-r^2}}{1-p^2} = \frac{\ell}{2(1-p^2)}\,, \quad 2p \le \ell \le 2 \tag{26}$$

Now from the theory of random planes, we know (cf. Kendall and Moran [6], pp. 20-22) that p has the uniform distribution. Hence the density of ℓ is

$$f(\ell) = \int_0^{\ell/2} f(\ell|p)dp = \int_0^{\ell/2} \frac{\ell}{2(1-p^2)}\,dp = \frac{\ell}{2}\int_0^{\ell/2}\left(\frac{1}{2(1+p)} + \frac{1}{2(1-p)}\right)dp$$

$$= \frac{\ell}{4}\Big[\ell n(1+p) - \log(1-p)\Big]_{p=0}^{p=\ell/2}$$

$$= \frac{\ell}{4}\,\ell n\,\frac{2+\ell}{2-\ell}\,, \quad 0 \le \ell \le 2 \tag{27}$$

$$E(\ell) = \frac{1}{4}\int_0^2 \ell^2 \ln\frac{2+\ell}{2-\ell}\,d\ell = \frac{1}{4}\left\{\int_0^2 \ell^2 \ln(2+\ell)\,d\ell - \int_0^2 \ell^2 \ln(2-\ell)\,d\ell\right\}$$

$$= \frac{1}{4}\left\{\int_{\ln 2}^{\ln 4} u(e^{3u}-4e^{2u}+4e^{u})\,du - \int_{-\infty}^{\ln 2} u(e^{3u}-4e^{2u}+4e^{u})\,du\right\}$$

$$= \frac{1}{4}\left\{\left[\frac{e^{3u}}{3}(u-\tfrac{1}{3})-4\frac{e^{2u}}{2}(u-\tfrac{1}{2})+4e^{u}(u-1)\right]_{u=\ln 2}^{u=\ln 4}\right.$$

$$\left.-\left[\frac{e^{3u}}{3}(u-\tfrac{1}{3})-4\frac{e^{2u}}{2}(u-\tfrac{1}{2})+4e^{u}(u-1)\right]_{\ln 2}^{\ln 4}\right\} = \frac{4}{3}\ln 2 + \frac{2}{3}\,.$$

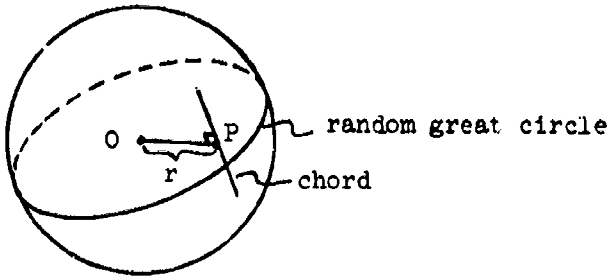
In a similar manner, we obtain $E(\ell^2) = \frac{8}{3}$.

Hence, $\operatorname{var}(\ell) = \frac{20}{9} - \frac{16}{9}\ln 2 - \frac{16}{9}(\ln 2)^2$.

<u>Model 8: Chord normal to plane of random great circle, through point uniformly distributed in circle of intersection.</u>

This randomness model corresponds to the measure obtained by taking the restriction of "random lines" in 3 dimensions to those intersecting the sphere, where "random lines" in 3 dimensions refers to the measure on lines which is invariant under translations, rotations and reflections.

Let r be the distance of the chord from the center.



By construction, P is uniformly distributed inside the great circle. Hence, applying (25) with $p = 0$, the density of r is

$$g(r) = 2r, \quad 0 \leq r \leq 1 \tag{28}$$

Making the transformation $\ell = 2\sqrt{1-r^2}$, we obtain the density of ℓ:

$$f(\ell) = \frac{\ell}{2}, \quad 0 \leq \ell \leq 2 \ . \tag{29}$$

We observe that this is the same density as that obtained under Model 1, where the chord is formed by joining two points independently uniformly distributed on the surface of the sphere. Since the measures on chords induced by both models are rotationally symmetric by construction, it is clear that the two measures are identical.

Model 8, being the measure induced on chords by "random lines" in 3 dimensions, could be considered the natural measure on chords of a sphere. This measure has the advantage of inducing the identical measure on chords of any sphere contained within the original sphere.

We have seen that this natural (or invariant) measure on chords can be obtained for the sphere by choosing the endpoints of the chord independently and uniformly on the surface. However, this is not true

in general for arbitrary convex figures in 3 dimensions. One way of constructing chords of an arbitrary convex body A in 3 dimensions having the natural measure is as follows: Choose the direction $\underset{\sim}{u}$ of the chord with probability proportional to the area of the projection of A along $\underset{\sim}{u}$. Let the intercept of the chord with the plane normal to $\underset{\sim}{u}$ be uniformly distributed over the projection of A onto the plane. This defines the chord uniquely.

Kingman [8] obtained relations between the measures induced under models 5,6 and 8 for arbitrary bounded convex regions in E^n. If we denote the length of the chord σ by $\ell(\sigma)$, and let μ_i be the measure on chords under model i, then his results were:

$$\mu_5(d\sigma) \propto \ell(\sigma)\ \mu_8(d\sigma) \tag{30}$$

$$\mu_6(d\sigma) \propto \ell^{n+1}(\sigma)\ \mu_8(d\sigma)\ . \tag{31}$$

Clearly, the densities we obtained for chord length in the case of a sphere in E^3 agree with these relations.

The following table summarizes our results.

Model	$f(\ell)$ $0 \leq \ell \leq 2$	$E(\ell)$	$Var(\ell)$
1. Joining two random points on surface	$\frac{\ell}{2}$	$\frac{4}{3} = 1.333$	$\frac{2}{9} = .222$
2. Random direction from random point on surface	$\frac{1}{2}$	$1 = 1.000$	$\frac{1}{3} = .333$
3. Random direction, uniformly distributed distance from center.	$\frac{\ell}{2\sqrt{4-\ell^2}}$	$\frac{\pi}{2} = 1.571$	$\frac{8}{3} - \frac{\pi^2}{4} = .199$
4. Chord center uniformly distributed inside sphere; random direction	$\frac{3}{8}\ell\sqrt{4-\ell^2}$	$\frac{3\pi}{8} = 1.178$	$\frac{8}{5} - \frac{9\pi^2}{64} = .212$
5. Random direction through random point inside sphere	$\frac{3}{8}\ell^2$	$\frac{3}{2} = 1.500$	$\frac{3}{20} = .150$
6. Through 2 random points inside sphere	$\frac{3}{32}\ell^5$	$\frac{12}{7} = 1.714$	$\frac{3}{49} = .061$
7. Normal to random intersecting plane, through random point in circle of intersection.	$\frac{\ell}{4}\ln\frac{2+\ell}{2-\ell}$	$\frac{4}{3}\ln 2 + \frac{2}{3} = 1.591$	$\frac{20}{9} - \frac{16}{9}(\ln 2 + \ln^2 2) = .134$
8. Normal to plane of random great circle, through random point in circle of intersection.	$\frac{\ell}{2}$	same as	Model 1

REFERENCES

[1] Bate, A. E. and Pillow, M. E. (1947). Mean free path of sound in an auditorium. Proc. Phys. Soc., 59, 535-541.

[2] Coleman, R. (1969). Random paths through convex bodies. J. Appl. Prob., 6, 430-441.

[3] David, F. N. and Fix, E. (1964). Intersections of random chords of a circle. Biometrika, 51, 373-379.

[4] Horowitz, M. (1965). Probability of random paths across elementary geometrical shapes. J. Appl. Prob., 2, 169-177.

[5] Jäger, G. (1911). Zur theorie des nachhalls. Sitzungsebr. Wiener Akad. Math. Naturw. Kl., 120, IIa, 613-634.

[6] Kendall, M. G. and Moran, P. A. P. (1963). Geometrical Probability. Griffin, London.

[7] Kingman, J. F. C. (1965). Mean free paths in a convex reflecting region. J. Appl. Prob., 2, 162-168.

[8] Kingman, J. F. C. (1969). Random secants of a convex body. J. Appl. Prob., 6, 660-672.